PERGAMON

International Journal of Heat and Mass Transfer 42 (1999) 345–356

International Journal of HEAT and MASS TRANSFER

The determination of two heat sources in an inverse heat conduction problem

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Received 7 November 1997

Abstract

There are some restrictions in the two sources estimation problem in recent studies. One of the restrictions is that the estimated results are inaccurate when two sources have different shapes and close distance. Another is that the accuracy of the estimation is questioned when the duration of two heat sources has a significant difference. The third restriction is that the estimation becomes inaccurate when the ratio of the peak values of the two heat sources is too large. Therefore, it is necessary to develop a robust method to estimate the strengths of two heat sources in order to alleviate the problems in past research. In this paper, a numerical algorithm coupled with the concept of future time is proposed to determine the problem sequentially. A special feature about this method is that no preselect functional form for the unknown sources is necessary and no sensitivity analysis is needed in the algorithm. Three examples are used to demonstrate the characteristics of the proposed method. From the results, they show that the proposed method is an accurate and efficient method to determine the strength of the two sources in the inverse heat conduction problems. © 1998 Elsevier Science Ltd. All rights reserved.

1. Introduction

The inverse source problem is the determination of the strength of the heat source from the temperature measured at a different point other than the sources' locations. It is an ill-posed problem because a small measurement error induces a large estimated error [1–9]. The inverse source problem is practical in many design and manufacturing areas in which the strength of the heat source is undetermined. Common problems include the detection of the quantity of the energy generation in a computer chip, or in a microwave heating process, or in a chemical reaction process.

The one-dimensional inverse problem with one unknown source has been investigated and satisfied results are reported [10, 11]. The problem with two unknown sources was also investigated by Silva Neto and Ozisik [12], but some limitations were presented. In past research, the estimated results were inaccurate when two sources had different shapes and close distance (ten per cent of the medium length). The accuracy of the estimation was influenced by a significantly different source strength duration between two heat sources, and the estimation became poor when the ratio of the peak values of the two heat sources was larger than six. Consequently, the application of the inverse technique to estimate two sources falls within a very limited scope. Therefore, a more robust algorithm needs to be developed in order to extend the scope of the application in the two source estimation problems.

The purpose of this research is to propose a sequential method to estimate the strength of two heat sources based on numerical computation which is more efficient than that of the symbolic computation [11]. Meanwhile, it also alleviates the restriction of the problems concluded in past research [12]. In the proposed method, there is no prior information on the functional form of the strength of the heat sources and there is no sensitivity analysis in the proposed algorithm. In the process of the derivation, a finite-element-difference method [13, 14] combined with

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0017–9310/98 $19.00
PII: S0017-9310(98)00128-8

the concept of future time [4] is used to derive the result. Then, the strength of two heat sources are determined step by step along with the temporal coordinate.

This paper includes five sections. In the present section, the current development of the inverse source problems is introduced and the features of the proposed method are also stated. In the second section, the characteristics of the inverse problem are delineated and the process of the proposed method is illustrated. In the third section, a computational algorithm is proposed to implement the method using a computer. In the fourth section, three examples are employed to demonstrate the process of the proposed method. A discussion of the analyzed results is also presented in this section. Finally, the overall contribution and possible applications of this research to the field of inverse heat conduction problems are concluded in the fifth section.

2. Approach to the two source estimation problem based on the proposed method

2.1. Problem statement

The inverse problem consists of finding the strength of two heat sources at different interior points of the spatial interval while the temperature measurements at the boundaries are given. Consider a slab with $\bar{L}$ thickness and constant thermal properties. This slab originally has a uniformly distributed temperature. At a specific time $\bar{t} = 0$, two heat sources $\bar{G}_1(\bar{t})$ and $\bar{G}_2(\bar{t})$ are applied to the interiors of the slab $\bar{x} = \bar{x}_{i1}$ and $\bar{x} = \bar{x}_{i2}$ while the front and back surfaces are adiabatic. Then, a dimensionless mathematical formation of the heat conduction problem is presented as follows:

$$\frac{\partial^2 T}{\partial x^2} + G_1(t)\delta(x - x_{i1}) + G_2(t)\delta(x - x_{i2}) = \frac{\partial T}{\partial t} \quad 0 < x < 1, \quad t > 0 \tag{1}$$

$$T(x, 0) = 0 \quad 0 \leqslant x \leqslant 1 \tag{2}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad t > 0 \tag{3}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=1} = 0 \quad t > 0 \tag{4}$$

where the following dimensionless quantities are defined as:

$$x = \frac{\bar{x}}{\bar{L}} \quad x_{i1} = \frac{\bar{x}_{i1}}{\bar{L}} \quad x_{i2} = \frac{\bar{x}_{i2}}{\bar{L}} \quad k = \frac{\bar{k}}{\bar{k}_r} \quad T = \frac{\bar{T}}{\bar{T}_0}$$

$$G_1(t) = \frac{\bar{L}^2}{\bar{k}}\frac{\bar{G}_1(\bar{t})}{\bar{T}_0} \quad G_2(t) = \frac{\bar{L}^2}{\bar{k}}\frac{\bar{G}_2(\bar{t})}{\bar{T}_0} \quad t = \frac{\bar{k}}{\overline{\rho C_{\mathrm{p}}}}\frac{\bar{t}}{\bar{L}^2}$$

$\bar{k}$ is the thermal conductivity and $\overline{\rho C_{\mathrm{p}}}$ is the heat capacity per unit volume, $\bar{T}_0$ and $\bar{G}_r$ refer to the nonzero reference temperature and the strength of the source, respectively. We assume $\bar{k} = \bar{k}_r$.

The inverse problem is given the temperature measured at $x = 0$ and $x = 1$ to estimate the strengths of the heat sources $G_1(t)$ and $G_2(\bar{t})$.

2.2. The method to determine the strength of two heat sources

The proposed method uses a finite-element method with linear element to discretize the spatial coordinate and uses a finite-difference method to discretize the temporal coordinate. A finite-element method with p equidistant grid at $t = t_j$ [14] is used to construct the following matrix equation:

$$[N]\{\dot{T}_j\} = \{S_j\} - [M]\{T_j\} \tag{5}$$

where

$$[M] = \begin{bmatrix} \frac{1}{\Delta x} & -\frac{1}{\Delta x} & \cdots & 0 & 0 \\ -\frac{1}{\Delta x} & \frac{2}{\Delta x} & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & -\frac{1}{\Delta x} & \frac{2}{\Delta x} & -\frac{1}{\Delta x} \\ 0 & \cdots & 0 & -\frac{1}{\Delta x} & \frac{1}{\Delta x} \end{bmatrix}_{(p+1)\times(p+1)} \tag{6}$$

$$[N] = \begin{bmatrix} \frac{\Delta x}{3} & \frac{\Delta x}{6} & \cdots & 0 & 0 \\ \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\ 0 & \cdots & 0 & \frac{\Delta x}{6} & \frac{\Delta x}{3} \end{bmatrix}_{(p+1)\times(p+1)} \tag{7}$$

$$\{S_j\} = \begin{Bmatrix} 0 \\ \cdots \\ \phi_j \\ \cdots \\ \varphi_j \\ \cdots \\ 0 \end{Bmatrix}_{(p+1)} \qquad \{\dot{T}_j\} = \begin{Bmatrix} \dot{T}_j^1 \\ \dot{T}_j^2 \\ \cdots \\ \dot{T}_j^p \\ \dot{T}_j^{p+1} \end{Bmatrix}_{(p+1)}$$

$$\{T_j\} = \begin{Bmatrix} T_j^1 \\ T_j^2 \\ \cdots \\ T_j^p \\ T_j^{p+1} \end{Bmatrix}_{(p+1)} \tag{8}$$

where $\phi_j = G_1(t_j)$, $\varphi_j = G_2(t_j)$, and $\dot{T}_j^i = (\mathrm{d}T_j^i/\mathrm{d}t)$. Δx is the increment of the spatial coordinate. The superscript is denoted as the index of the spatial grid and the subscript is denoted as the index of the temporal grid. The locations of ϕ_j and φ_j in vector $\{S_j\}$ are at the grids corresponding with the source locations x_{i1} and x_{i2}.

Next, we consider our finite element expression for $\{\dot{T}_j\}$ as a backward difference at time t_j. Therefore, we have

$$\{\dot{T}_j\} = \frac{1}{\Delta t}\{T_j\} - \frac{1}{\Delta t}\{T_{j-1}\} \tag{9}$$

Here Δt is the increment of the temporal coordinate.

Substitute equation (9) into equation (5), we have the following differential equation:

$$[K]\{T_j\} = [B]\{T_{j-1}\} + \{S_j\} \tag{10}$$

where

$$[K] = [M] + \frac{1}{\Delta t}[N] \quad \text{and} \quad [B] = \frac{1}{\Delta t}[N]$$

When $t = t_j$, the temperature distribution $\{T_j\}$ can be derived from equation (10) as follows:

$$\begin{aligned} \{T_j\} &= [K]^{-1}[B]\{T_{j-1}\} + [K]^{-1}\{S_j\} \\ &= [C]\{T_{j-1}\} + [D]\{S_j\} \end{aligned} \tag{11}$$

where $[C] = [K]^{-1}[B]$ and $[D] = [K]^{-1}$.

Similarly, the temperature distribution at $t = t_m, t_{m+1}, \ldots, t_{m+r-1}$ can be represented as follows:

$$\begin{aligned} \{T_m\} &= [C]\{T_{m-1}\} + [D]\{S_m\} \\ \{T_{m+1}\} &= [C]\{T_m\} + [D]\{S_{m+1}\} \\ &= [C]^2\{T_{m-1}\} + [C][D]\{S_m\} + [D]\{S_{m+1}\} \\ &\cdots \\ \{T_{m+r-1}\} &= [C]\{T_{m+r-2}\} + [D]\{S_{m+r-1}\} \\ &= [C]^r\{T_{m-1}\} \\ &\quad + [C]^{r-1}[D]\{S_m\} + [C]^{r-2}[D]\{S_{m+1}\} + \cdots \\ &\quad + [C][D]\{S_{m+r-2}\} + [D]\{S_{m+r-1}\} \end{aligned} \tag{12}$$

The vector $\{S\}$ is composited by the unknown heat sources ϕ and φ (i.e., $\{S_j\} = \{u^{i1}\}\phi_j + \{u^{i2}\}\varphi_j$). $\{u^{i1}\}$ and $\{u^{i2}\}$ are the unit column vectors with a unit at $i1$th and $i2$th component, respectively. As well, $i1$ and $i2$ are the grid number of the locations of the estimated function ϕ_j and φ_j, respectively. And then a unit row vector $\lfloor u^i \rfloor$ (*i.e.*, a unit at i-component) times to both sides of equation (12), the temperature at i-spatial grid can be calculated as:

$$\begin{aligned} T_m^i &= \lfloor u^i \rfloor\{T_m\} = \lfloor u^i \rfloor[C]\{T_{m-1}\} \\ &\quad + \lfloor u^i \rfloor[D]\{u^{i1}\}\phi_m + \lfloor u^i \rfloor[D]\{u^{i2}\}\varphi_m \\ T_{m+1}^i &= \lfloor u^i \rfloor\{T_{m+1}\} = \lfloor u^i \rfloor[C]\{T_m\} \\ &\quad + \lfloor u^i \rfloor[D]\{u^{i1}\}\phi_{m+1} + \lfloor u^i \rfloor[D]\{u^{i2}\}\varphi_{m+1} \\ &= \lfloor u^i \rfloor[C]^2\{T_{m-1}\} + \lfloor u^i \rfloor[C][D]\{u^{i1}\}\phi_m \\ &\quad + \lfloor u^i \rfloor[D]\{u^{i1}\}\phi_{m+1} \\ &\quad + \lfloor u^i \rfloor[C][D]\{u^{i2}\}\varphi_m + \lfloor u^i \rfloor[D]\{u^{i2}\}\varphi_{m+1} \\ &\cdots \\ T_{m+r-1}^i &= \lfloor u^i \rfloor\{T_{m+r-1}\} \\ &= \lfloor u^i \rfloor[C]\{T_{m+r-2}\} + \lfloor u^i \rfloor[D]\{u^{i1}\}\phi_{m+r-1} \\ &\quad + \lfloor u^i \rfloor[D]\{u^{i2}\}\varphi_{m+r-1} \\ &= \lfloor u^i \rfloor[C]^r\{T_{m-1}\} + \lfloor u^i \rfloor[C]^{r-1}[D]\{u^{i1}\}\phi_m \\ &\quad + \lfloor u^i \rfloor[C]^{r-2}[D]\{u^{i1}\}\phi_{m+1} + \cdots \\ &\quad + \lfloor u^i \rfloor[C][D]\{u^{i1}\}\phi_{m+r-2} + \lfloor u^i \rfloor[D]\{u^{i1}\}\phi_{m+r-1} \\ &\quad + \lfloor u^i \rfloor[C]^{r-1}[D]\{u^{i2}\}\varphi_m \\ &\quad + \lfloor u^i \rfloor[C]^{r-2}[D]\{u^{i2}\}\varphi_{m+1} + \cdots \\ &\quad + \lfloor u^i \rfloor[C][D]\{u^{i2}\}\varphi_{m+r-2} + \lfloor u^i \rfloor[D]\{u^{i2}\}\varphi_{m+r-1} \end{aligned} \tag{13}$$

Then, the temperatures at i-spatial grid can be expressed as follows:

$$\begin{aligned} T_m^i &= a_{m,0}^i + a_{m,m}^{i,i1}\phi_m + a_{m,m}^{i,i2}\varphi_m \\ T_{m+1}^i &= a_{m+1,0}^i + a_{m+1,m}^{i,i1}\phi_m + a_{m+1,m+1}^{i,i1}\phi_{m+1} \\ &\quad + a_{m+1,m}^{i,i2}\varphi_m + a_{m+1,m+1}^{i,i2}\varphi_{m+1} \\ T_{m+2}^i &= a_{m+2,0}^i + a_{m+2,m}^{i,i1}\phi_m \\ &\quad + a_{m+2,m+1}^{i,i1}\phi_{m+1} + a_{m+2,m+2}^{i,i1}\phi_{m+2} \\ &\quad + a_{m+2,m}^{i,i2}\varphi_m + a_{m+2,m+1}^{i,i2}\varphi_{m+1} \\ &\quad + a_{m+2,m+2}^{i,i2}\varphi_{m+2} \\ &\cdots \\ T_{m+r-1}^i &= a_{m+r-1,0}^i + a_{m+r-1,m}^{i,i1}\phi_m \\ &\quad + a_{m+r-1,m+1}^{i,i1}\phi_{m+1} + a_{m+r-1,m+2}^{i,i1}\phi_{m+2} + \cdots \\ &\quad + a_{m+r-1,m+r-2}^{i,i1}\phi_{m+r-2} + a_{m+r-1,m+r-1}^{i,i1}\phi_{m+r-1} \\ &\quad + a_{m+r-1,m}^{i,i2}\varphi_m + a_{m+r-1,m+1}^{i,i2}\varphi_{m+1} \\ &\quad + a_{m+r-1,m+2}^{i,i2}\varphi_{m+2} + \cdots \\ &\quad + a_{m+r-1,m+r-2}^{i,i2}\varphi_{m+r-2} + a_{m+r-1,m+r-1}^{i,i2}\varphi_{m+r-1} \end{aligned} \tag{14}$$

where

$$\begin{aligned} a_{m,m}^{i,i^*} &= a_{m+1,m+1}^{i,i^*} = \cdots = a_{m+r-1,m+r-1}^{i,j^*} \\ &= \lfloor u^i \rfloor[D]\{u^i\} = e_0^{i,j^*} \\ a_{m+1,m}^{i,i^*} &= a_{m+2,m+1}^{i,i^*} = \cdots = a_{m+r-1,m+r-2}^{i,i^*} \end{aligned}$$

$$= \lfloor u^i \rfloor [C][D]\{u^{i^*}\} = e_1^{i,i^*}$$

$$a_{m+2,m}^{i,i^*} = a_{m+3,m+1}^{i,i^*} = \cdots = a_{m+r-1,m+r-3}^{i,i^*} = \lfloor u^i \rfloor [C]^2[D]\{u^{i^*}\} = e_2^{i,i^*}$$
$$\ldots \tag{15}$$
$$a_{m+r-1,m}^{i,i^*} = \lfloor u^i \rfloor [C]^{r-1}[D]\{u^{i^*}\} = e_{r-1}^{i,i^*}$$

Here, the subscript of e is the difference of the subscripts of a. The superscripts i and i^* are the grid numbers of the measured location and the source location, respectively.

As well, the following properties can be derived,

$$a_{m,0}^i = \lfloor u^i \rfloor [C]\{T_{m-1}\}$$
$$a_{m+1,0}^i = \lfloor u^i \rfloor [C]^2\{T_{m-1}\}$$
$$a_{m+2,0}^i = \lfloor u^i \rfloor [C]^3\{T_{m-1}\}$$
$$\ldots$$
$$a_{m+r-1,0}^i = \lfloor u^i \rfloor [C]^r\{T_{m-1}\} \tag{16}$$

where $\{T_{m-1}\}$ is the temperature distribution at $t = t_{m-1}$. The value of i indicates the grid number of the measured location. In this study, the values of j are 1 and $p+1$.

The coefficient $a_{m,m}^{i,i1}, a_{m+1,m}^{i,i1}, \ldots, a_{m+r-1,m}^{i,i1}, a_{m,m}^{i,i2}, \ldots$, and $a_{m+r-1,m}^{i,i2}$ in equation (15) depends on the locations of the measured point and the input source strengths. As well, the values of the coefficients count on the step number of future time but not the time step in the global temporal coordinate. In other words, those coefficients are constant in each iteration and they only need to be calculated once when the locations of the measured points and the input sources are fixed. It is different from the symboic method developed in Ref. [11], which needs to calculate the coefficients e_0^{i,i^*}, e_1^{i,i^*}, ..., and e_{r-1}^{i,i^*} in each iteration. On the other hand, the coefficients in equation (16) $a_{m,0}^i$, $a_{m+1,0}^i$, $a_{m+2,0}^i, \ldots$, and $a_{m+r-1,0}^i$ are derived from matrix $[C]$ and the previous state $\{T_{m-1}\}$. Therefore,these coefficients need to be evaluated iteratively.

When $t = t_m$, the estimated condition, ϕ_1, ϕ_2, $\phi_3, \ldots, \phi_{m-1}$, φ_1, φ_2, φ_3, ..., and φ_{m-1} have been evaluated the problem is to estimate the strength of the heat sources ϕ_m and φ_m. To stabilize the estimated results in the inverse algorithms, the sequential procedure is assumed temporally that several future source strengths are constant [4]. Then, the unknown conditions in the future time are assumed to be equal, i.e.,

$$\phi_{m+1} = \phi_{m+2} = \cdots = \phi_{m+r-2} = \phi_{m+r-1} = \phi_m \tag{17}$$

$$\varphi_{m+1} = \varphi_{m+2} = \cdots = \varphi_{m+r-2} = \varphi_{m+r-1} = \varphi_m \tag{18}$$

Here r is the number of the future time.

With the assumption of the future time, equation (18) can be rewritten to the following form:

$$T_m^i = a_{m,0}^i + a_{m,m}^{i,i1}\phi_m + a_{m,m}^{i,i2}\varphi_m = a_{m,0}^i + e_0^{i,i1}\phi_m + e_0^{i,i2}\varphi_m$$

$$\begin{aligned}
T_{m+1}^i &= a_{m+1,0}^i + (a_{m+1,m}^{i,i1} + a_{m+1,m+1}^{i,i1})\phi_m \\
&\quad + (a_{m+1,m}^{i,i2} + a_{m+1,m+1}^{i,i2})\varphi_m \\
&= a_{m+1,0}^i + (e_1^{i,i1} + e_0^{i,i1})\phi_m + (e_1^{i,i2} + e_0^{i,i2})\varphi_m \\
T_{m+2}^i &= a_{m+2,0}^i + (a_{m+2,m}^{i,i1} + a_{m+2,m+1}^{i,i1} + a_{m+2,m+2}^{i,i1})\phi_m \\
&\quad + (a_{m+2,m}^{i,i2} + a_{m+2,m+1}^{i,i2} + a_{m+2,m+2}^{i,i2})\varphi_m \\
&= a_{m+2,0}^i + (e_2^{i,i1} + e_1^{i,i1} + e_0^{i,i1})\phi_m \\
&\quad + (e_2^{i,i2} + e_1^{i,i2} + e_0^{i,i2})\varphi_m \\
&\quad \ldots \\
T_{m+r-1}^i &= a_{m+r-1,0}^i + (a_{m+r-1,m}^{i,i1} + a_{m+r-1,m+1}^{i,i1} \\
&\quad + a_{m+r-1,m+2}^{i,i1} + \cdots \\
&\quad + a_{m+r-1,m+r-2}^{i,i1} + a_{m+r-1,m+r-1}^{i,i1})\phi_m \\
&\quad + (a_{m+r-1,m}^{i,i2} + a_{m+r-1,m+1}^{i,i2} + a_{m+r-1,m+2}^{i,i2} + \cdots \\
&\quad + a_{m+r-1,m+r-2}^{i,i2} + a_{m+r-1,m+r-1}^{i,i2})\varphi_m \\
&= a_{m+r-1,0}^i + (e_{r-1}^{i,i1} + e_{r-2}^{i,i1} + e_{r-3}^{i,i1} \\
&\quad + \cdots + e_1^{i,i1} + e_0^{i,i1})\phi_m \\
&\quad + (e_{r-1}^{i,i2} + e_{r-2}^{i,i2} + e_{r-3}^{i,i2} + \cdots + e_1^{i,i2} + e_0^{i,i2})\varphi_m
\end{aligned} \tag{19}$$

We define

$$T_{m+k}^i = a_{m+k,0}^i + E_k^{i,i1}\phi_m + E_k^{i,i2}\varphi_m = a_{m+k,0}^i + \lfloor E_k^{i,i1} \quad E_k^{i,i2} \rfloor \begin{Bmatrix} \phi_m \\ \varphi_m \end{Bmatrix} \tag{20}$$

where

$$E_k^{i,i^*} = \sum_{l=0}^{k} e_l^{i,i^*} \quad \text{and} \quad k = 0, 1, 2, \ldots, r-1$$

and i is the grid number of the measured location and i^* is the grid number of the source location.

$$\boldsymbol{\vartheta} = \boldsymbol{\Phi\theta} \tag{21}$$

where

$$\boldsymbol{\vartheta} = \begin{Bmatrix} T_m^{i_1} - a_{m,0}^{i_1} \\ T_{m+1}^{i_1} - a_{m+1,0}^{i_1} \\ T_{m+2}^{i_1} - a_{m+2,0}^{i_1} \\ \ldots \\ T_{m+r-1}^{i_1} - a_{m+r-1,0}^{i_1} \\ T_m^{i_2} - a_{m,0}^{i_2} \\ T_{m+1}^{i_2} - a_{m+1,0}^{i_2} \\ T_{m+2}^{i_2} - a_{m+2,0}^{i_2} \\ \ldots \\ T_{m+r-1}^{i_2} - a_{m+r-1,0}^{i_2} \end{Bmatrix}_{2r\times 1}$$

$$\mathbf{\Phi} = \begin{Bmatrix} E_0^{i_1,i1} & E_0^{i_1,i2} \\ E_1^{i_1,i1} & E_1^{i_1,i2} \\ E_2^{i_1,i1} & E_2^{i_1,i2} \\ \cdots & \cdots \\ E_{r-1}^{i_1,i1} & E_{r-1}^{i_1,i2} \\ E_0^{i_2,i1} & E_0^{i_2,i2} \\ E_1^{i_2,i1} & E_1^{i_2,i2} \\ E_2^{i_2,i1} & E_2^{i_2,i2} \\ \cdots & \cdots \\ E_{r-1}^{i_2,i1} & E_{r-1}^{i_2,i2} \end{Bmatrix}_{2r\times 2} \qquad \theta = \begin{Bmatrix} \phi_m \\ \varphi_m \end{Bmatrix} \tag{22}$$

After the measured temperature Y_j^i (measured at $t = t_j$ and $x = x_i$) is substituted into vector $\boldsymbol{\vartheta}$, the components of vector $\boldsymbol{\theta}$ can be found through a linear least-squares error method [15]. Therefore, the result is:

$$\hat{\boldsymbol{\theta}} = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\boldsymbol{\vartheta} \tag{23}$$

This equation provides a sequential algorithm that can be used to estimate the two sources through increasing the value of m by one for each time step. Therefore, the strength of the heat sources ϕ and φ can be solved iteratively along the temporal coordinate. The proposed method is based on the finite-element-difference approach, and it can be extended to use other kinds of numerical methods through the proposed algorithm.

3. Computational algorithm

The procedure for the proposed method can be summarized as follows: first, we choose the number of future time r, the discretized spatial size Δx and temporal size Δt, and the measured grids $i1$ and $i2$. Then, matrix $[B]$ and stiffness matrix $[K]$ of the finite-element model are known. Thus, matrices $[C]$, $[C]^2, \ldots, [C]^r$ and $[D]$, $[C][D], \ldots, [C]^{r-1}[D]$ can be calculated in advance and the coefficients $E_k^{i,i1}$ and $E_k^{i,i2}$ are known before the iteration. After that, the iteration includes the following steps:

Step 1. Let $j = m$ and the temperature distribution at $\{T_{j-1}\}$ is known.
Step 2. Calculate $a_{j,0}^i$, $a_{j+1,0}^i$, $a_{j+2,0}^i, \ldots,$ and $a_{j+r-1,0}^i$ through equation (16).
Step 3. Collect the measurement at boundaries Y_j, $Y_{j+1}, \ldots, Y_{j+r-1}$.
Step 4. Calculate $\hat{\boldsymbol{\theta}} = \lfloor \hat{\phi}_j \hat{\varphi}_j \rfloor^T$ according to equation (23).
Step 5. Calculate $\{T_j\}$.
Step 6. Terminate the process if the final iteration is attached. Otherwise let $j = m+1$ return to Step 2.

4. Results and discussion

In this section, problems defined from equations (1)–(4) are used as examples to estimate the strength of the heat sources. The strength of the heat sources is chosen with a step change or a sharp corner over temporal coordinate because those kinds of forms are most difficult to recover in the inverse analysis. Three examples are used to demonstrate the proposed method that can estimate the strength of two sources accurately even though some restrictions exist. The exact temperature and source terms used in the following examples are preselected so that these functions can satisfy equations (1)–(6). The accuracy of the proposed method is assessed by comparing the estimated strengths of the heat sources with the preselected heat sources. Meanwhile, the simulated temperature measurement is generated from the exact temperature in each problem and it is presumed to have measurement errors. In other words, the random errors of measurement are added to the exact temperature. It can be shown in the following equation:

$$Y_j^i = T_j^i + \lambda_{i,j}\sigma \tag{24}$$

where the subscripts i and j are the grid number of spatial-coordinate and temporal-coordinate, respectively. T_j^i in equation (24) is the exact temperature. Y_j^i is the measured temperature. σ is the standard deviation of measurement errors. $\lambda_{i,j}$ is a random number. The value of $\lambda_{i,j}$ is calculated by the IMSL subroutine DRNNOR [16] and chosen over the range $-2.576 < \lambda_{i,j} < 2.576$, which represents the 99% confidence bound for the measurement temperature.

The temporal domain is from 0.02–2 with 0.02 increment for the example problems. Two thermocouples are allocated at the front and back surfaces of the medium. Detailed descriptions for the examples are shown as follows:

Example 1: Two heat sources $G_1(t)$ and $G_2(t)$ have different shapes and are placed closed. Two cases $(x_{i1}, x_{i2}) = (0.4, 0.5)$ and $(0.49, 0.51)$ are discussed. The heat sources $G_1(t)$ and $G_2(t)$ are presumed in the following forms:

$$G_1(t) = 0 \quad t \leqslant 0.4 \quad \text{or } t \geqslant 1.6$$

$$G_1(t) = 0.5 \quad 0.4 < t < 1.6$$

and

$$G_2(t) = 0.3 \quad t \leqslant 0.4$$

$$G_2(t) = \tfrac{1}{3}(t-1) + 0.5 \quad 0.4 < t \leqslant 1$$

$$G_2(t) = -\tfrac{2}{3}(t-1) + 0.5 \quad 1 < t \leqslant 1.6$$

$$G_2(t) = 0.1 \quad t > 1.6$$

In the first case $(x_{i1}, x_{i2}) = (0.4, 0.5)$, the estimated results are not accurate in Silva Neto and Ozisik's approach. They concluded that it is not possible to estimate the source strengths by the measurements taken at both boundary surfaces when the two sources are very close to each other and have distinctly different shapes. However, the estimated results of the proposed method shown in Fig. 1a and b are acceptable. Furthermore, the estimated results (see Fig. 1c and d) are still good when two sources get closer [i.e., $(x_{i1}, x_{i2}) = (0.49, 0.51)$]. Therefore, the proposed method can estimate two distinctly different shapes sources accurately even though they are close to each other.

Example 2: The effect of a different strength duration between two heat sources is examined. The source locations are $x_{i1} = 0.1$ and $x_{i2} = 0.9$. The source strength $G_1(t)$ and $G_2(t)$ are presumed in the following forms:

$G_1(t) = 0 \quad t \leqslant 0.4 \quad \text{or } t \geqslant 1.6$

$G_1(t) = 0.5 \quad 0.4 < t < 1.6$

$G_2(t) = 0 \quad t \leqslant d_1 \quad \text{or } t \geqslant d_2$

$G_2(t) = 0.5 \quad d_1 < t < d_2$

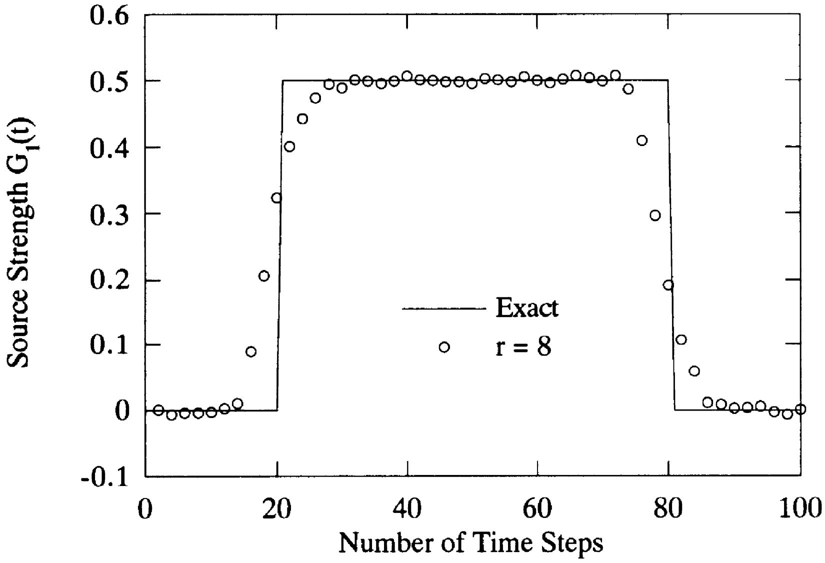

Fig. 1a. The estimation of the strength of $G_1(t)$ with $\sigma = 0.01$ in example one when two sources are located at $x_{i1} = 0.4$ and $x_{i2} = 0.5$.

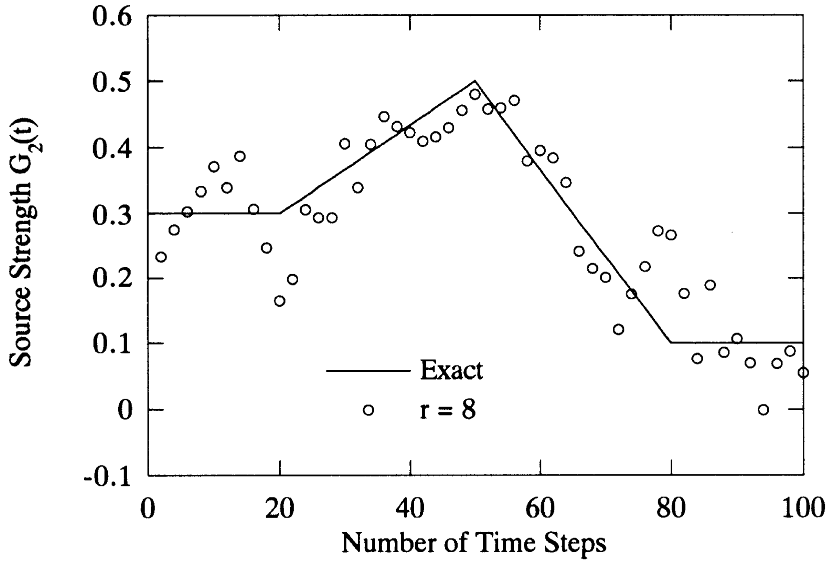

Fig. 1b. The estimation of the strength of $G_2(t)$ with $\sigma = 0.01$ in example one when two sources are located at $x_{i1} = 0.4$ and $x_{i2} = 0.5$.

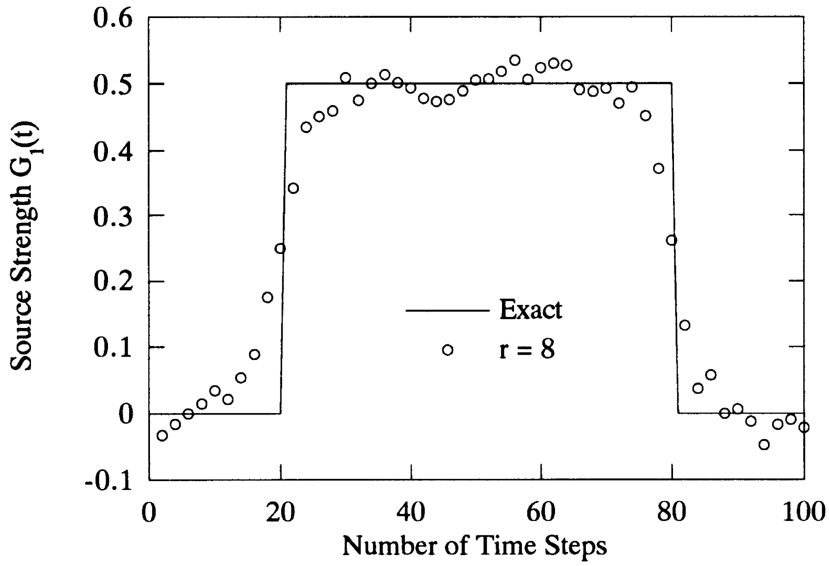

Fig. 1c. The estimation of the strength of $G_1(t)$ with $\sigma = 0.01$ in example one when two sources are located at $x_{i1} = 0.49$ and $x_{i2} = 0.51$.

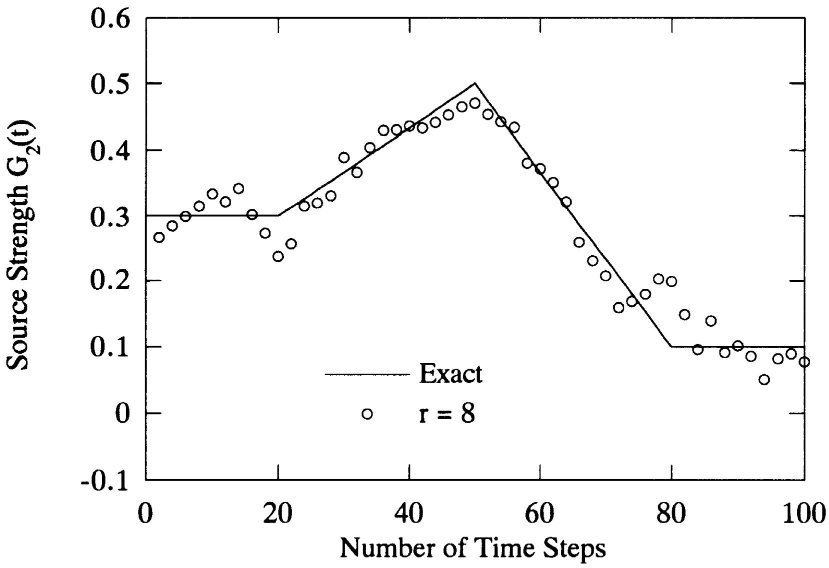

Fig. 1d. The estimation of the strength of $G_2(t)$ with $\sigma = 0.01$ in example one when two sources are located at $x_{i1} = 0.49$ and $x_{i2} = 0.51$.

Three cases are examined that are $(d_1, d_2) = (0.9, 1.0)$, $(0.8, 1.0)$, and $(0.7, 1.1)$.

In the second example, the effect of source duration is testified. When the problem with $(d_1, d_2) = (0.8, 1.0)$ is solved, the results show that the accuracy of the present estimation is better than that of Silva Neto and Ozisik's research (see Fig. 2a and b). Furthermore, the problem with various source duration is verified. The ratio of the duration of $G_1(t)$ and $G_2(t)$ is 12, 6, and 3, respectively. The means and variances of estimated error (i.e., estimated results minus exact function) are shown in Table 1. First, we compare the results of $G_2(t)$ when $(d_1, d_2) = (0.9, 1.0)$, $(0.8, 1.0)$ and $(0.7, 1.1)$. It is clear that the means of $G_2(t)$ in the different duration are very close to one another, and the variances of $G_2(t)$ are close to one another, too. Second, the stochastic parameters of $G_1(t)$ and $G_2(t)$ are observed when $(d_1, d_2) = (0.9, 1.0)$. The means are -0.00101826 and -0.00010812 for $G_1(t)$ and $G_2(t)$, respectively, and the variances are 0.00412058 and 0.00440229 for $G_1(t)$ and $G_2(t)$, respectively. The

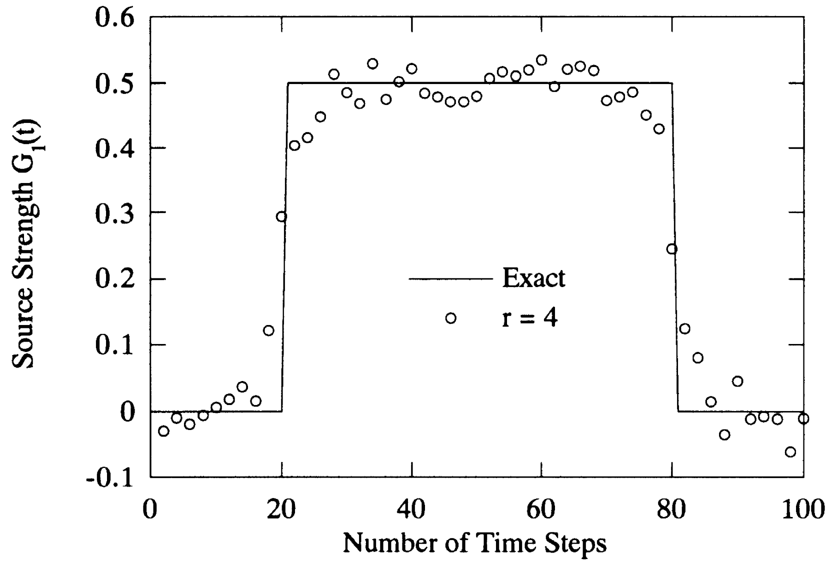

Fig. 2a. The estimation of the strength of $G_1(t)$ with $\sigma = 0.01$ in example two when two sources are located at $x_{i1} = 0.1$ and $x_{i2} = 0.9$.

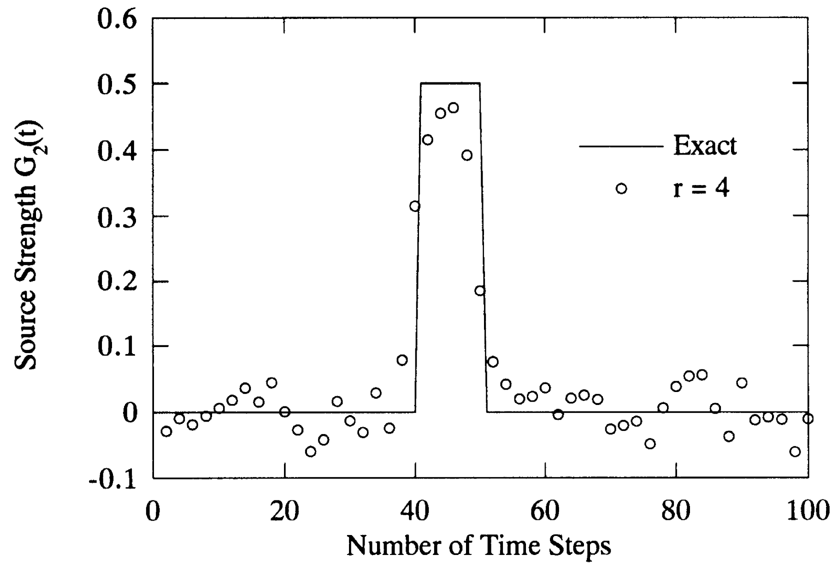

Fig. 2b. The estimation of the strength of $G_2(t)$ with $\sigma = 0.01$ in example two when two sources are located at $x_{i1} = 0.1$ and $x_{i2} = 0.9$.

Table 1
The means and variances of the error function in example two ($r = 4$)

	$G_1(t)$ Mean	$G_1(t)$ Variance	$G_2(t)$ Mean	$G_2(t)$ Variance
$d_1 = 0.9$, $d_2 = 1$	-0.00101826	0.00412058	-0.00101812	0.00440229
$d_1 = 0.8$, $d_2 = 1$	-0.00101827	0.00411222	-0.00101812	0.00447798
$d_1 = 0.7$, $d_2 = 1.1$	-0.00101837	0.00411286	-0.00101822	0.00415895

results show that the stochastic values are close to each other when $G_1(t)$ and $G_2(t)$ are estimated simultaneously. In conclusion, the results from the proposed method show that the source duration does not have a strong impact on the problem.

Example 3: The heat sources $G_1(t)$ and $G_2(t)$ have triangular shapes, and different peak strengths. The locations of the heat sources are $x_{i1} = 0.05$ and $x_{i2} = 0.95$. $G_1(t)$ and $G_2(t)$ have the following forms:

$$G_1(t) = 0 \quad G_2(t) = G_1(t) \quad t \leqslant 0.4 \quad \text{or } t \geqslant 1.6$$

$$G_1(t) = \frac{h}{1.2}(t-0.4) \quad G_2(t) = \frac{1}{1.2}(t-0.4) \quad 0.4 < t \leqslant 1$$

$$G_1(t) = \frac{h}{0.8}(t-1.6) \quad G_2(t) = \frac{1}{0.8}(t-1.6) \quad 1 < t < 1.6$$

where $h = 3$, 6, 9, 12, 24, 48, and 96.

In Silva Neto and Ozisik's approach, they concluded that the estimations were poor when the value of h is greater than six. In this research, the estimation of $G_1(t)$ is always good when $h = 3$, 6, 9, 12, 24, 48, and 96 (see Fig. 3a and b). However, the estimation of $G_2(t)$ gets worse when the value of h gets larger (see Fig. 3c). Therefore, a modification of the proposed method is needed to increase the accuracy of the estimation of $G_2(t)$. The modified method is first to execute the proposed method from which an accurate estimation of $G_1(t)$ can be obtained (see Fig. 3a and b). Then, the estimated value of $G_1(t)$ is substituted into the heat equation and leads to an inverse problem with one unknown source strength. Consequently, the estimated value of $G_2(t)$ is solved. From the estimated results, it shows that the modified method can estimate the value of $G_2(t)$ accurately when $h = 12$ (see Fig. 3d). Furthermore, the method is implemented to more restricted situation in which the value of h is 24, 48, and 96, respectively (see Fig. 3e). As well, the results are still acceptable.

In all examples, the estimated results match with the exact solutions when measurement errors are not included and the number of future time is one ($r = 1$). All numerical calculation is performed on a personal computer with a Pentium-133 CPU. The computation required about 2.37 s CPU time when the number of future time $r = 8$ is taken in example one, about 1.31 s CPU time when the number of future time $r = 4$ are taken in example two, and about 1.65 s CPU time when the modified method and the number of future time $r = 4$ is used in example three. However, Silva Neto and Ozisik spend about 10 s of CPU time in the CRAY Y-MP super computer in their approach. Therefore, the proposed method is a considerably faster inverse algorithm.

From the above discussion, it can be concluded that the proposed method is an accurate, robust and efficient method to determine the strength of two sources in the inversive heat conduction problems.

5. Conclusion

An efficient algorithm has been introduced for determining the strength of two sources in the inverse conduction problems. The inverse solution is represented as a closed form which is derived from a finite-difference-

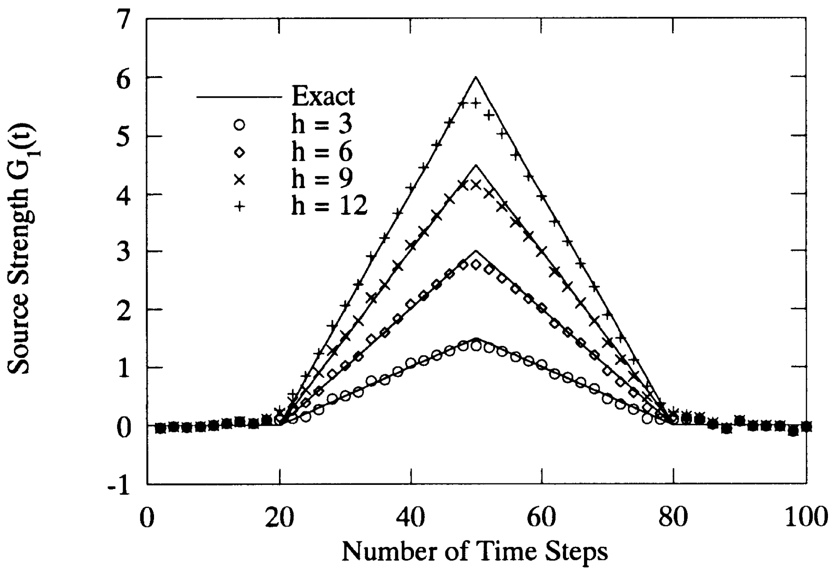

Fig. 3a. The estimation of the strength of $G_1(t)$ with $\sigma = 0.02$ in example three when two sources are located at $x_{i1} = 0.05$ and $x_{i2} = 0.95$ ($r = 4$ and $h = 3$, 6, 9, and 12).

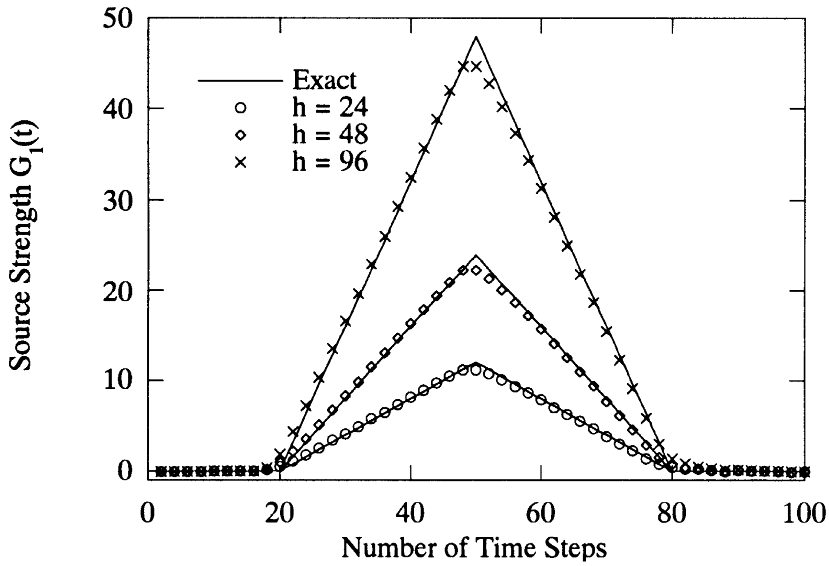

Fig. 3b. The estimation of the strength of $G_1(t)$ with $\sigma = 0.02$ in example three when two sources are located at $x_{i1} = 0.05$ and $x_{i2} = 0.95$ ($r = 4$ and $h = 24$, 48, and 96).

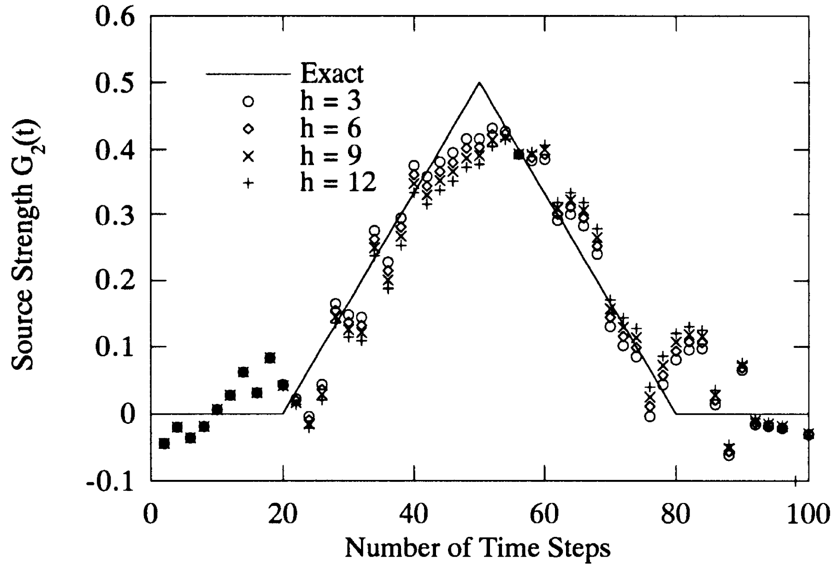

Fig. 3c. The estimation of the strength of $G_2(t)$ with $\sigma = 0.02$ in example three based on the proposed method when two sources are located at $x_{i1} = 0.05$ and $x_{i2} = 0.95$ ($r = 4$ and $h = 3$, 6, 9, and 12).

element method when the temperature measurements are available. A special feature of this method is that no preselect functional form for the unknown sources is necessary and no sensitivity analysis is needed in the algorithm. Three examples have been illustrated based on the proposed method. The result shows that the proposed method can estimate the strength of two sources accurately even though two sources have different shape and close distance, two sources have a significantly different strength duration, and two sources have a large value of the ratio of the peak values. In comparison with past research, the results show that the proposed method is

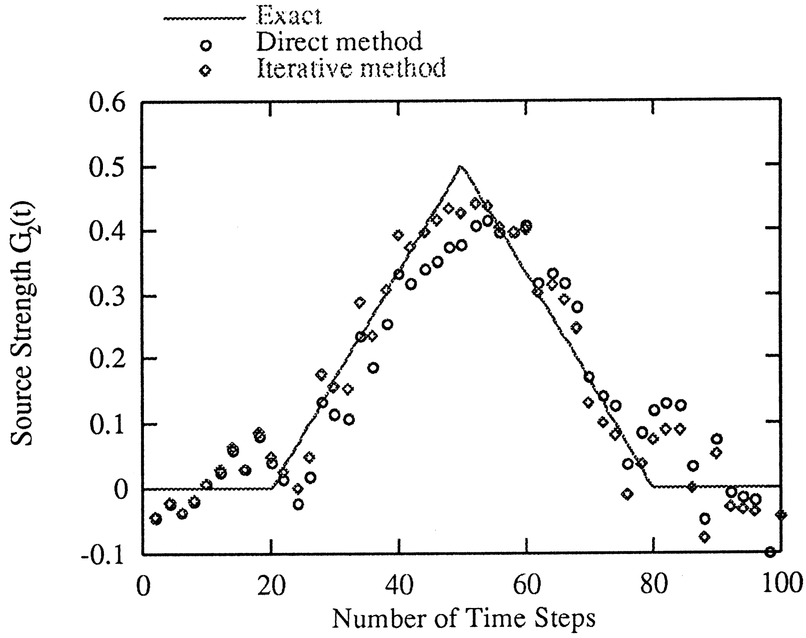

Fig. 3d. The comparison of the estimated results of $G_2(t)$ in example three based on the proposed method and the modified method ($x_{i1} = 0.05$ and $x_{i2} = 0.95$, $r = 4$, and $h = 12$).

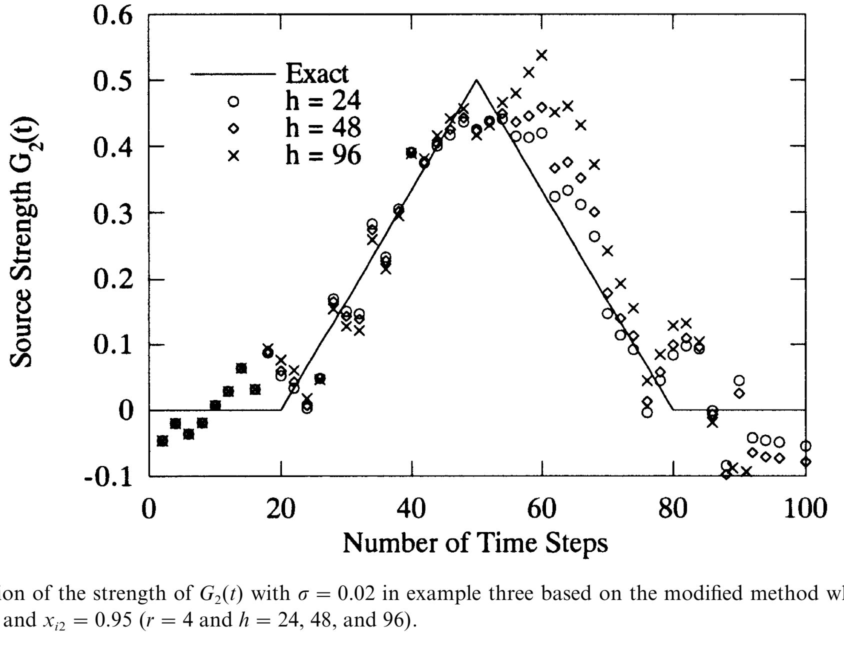

Fig. 3e. The estimation of the strength of $G_2(t)$ with $\sigma = 0.02$ in example three based on the modified method when two sources are located at $x_{i1} = 0.05$ and $x_{i2} = 0.95$ ($r = 4$ and $h = 24$, 48, and 96).

an accurate, robust, and efficient inverse technique. The proposed method is applicable to the other kinds of inverse problems such as boundary and source strength estimation in the multi-dimensional inverse conduction problems.

References

[1] G. Stolz, Jr., Numerical solutions to an inverse problem of heat conduction for simple shapes, ASME : Journal of Heat Transfer 82 (1960) 20–26.

[2] E.M. Sparrow, A. Haji-Sheikh, T.S. Lundgren, The inverse problem in transient heat conduction, ASME Journal of Applied Mechanics 86 (1964) 369–375.

[3] A.N. Tikhonov, V.Y. Arsenin, Solutions of Ill-posed Problems, Winston, Washington, DC, 1977.

[4] J.V. Beck, B. Blackwell, C.R. St. Clair, Inverse Heat Conduction—Ill-posed Problem, Wiley, New York, 1985.

[5] Y. Jarny, M.N. Ozisik, J.P. Bardon, A general optimization method using adjoint equation for solving multidimensional inverse heat conduction, International Journal of Heat and Mass Transfer 34 (11) (1991) 2911–2919.

[6] E. Hensel, Inverse Theory and Applications for Engineers, Prentice-Hall, Englewood Cliffs, NJ, 1991.

[7] V.A. Morozov, M. Stressin, Regularization Methods for Ill-posed Problems, CRC Press, Boca Raton, FL, 1993.

[8] D.A. Murio, The Mollification Method and the Numerical Solution of Ill-posed Problems, Wiley-Interscience, New York, 1993.

[9] O.M. Alifanov, Inverse Heat Transfer Problems, Springer-Verlag, Berlin, Heidelberg, 1994.

[10] C.H. Huang, M.N. Ozisik, Inverse problem of determining the unknown strength of an internal plate heat source, Journal of The Fanklin Institute 329 (4) (1992) 751–764.

[11] C.-Y. Yang, A sequential method to estimate the strength of the heat source based on symbolic computation, International Journal of Heat and Mass Transfer, 41 (14) (1998) 2245–2252.

[12] A.J. Silva Netto, M.N. Ozisik, Inverse problem of simultaneously estimating the time-varying strengths of two plane heat sources, Journal of Applied Physics 73 (5) (1993) 2132–2137.

[13] B. Carnahan, H.A. Luther, J.O. Wilkes, Applied Numerical Methods, 1st ed. John Wiley and Sons, New York, 1997.

[14] M.A. Celia, W.G. Gray, Numerical Methods for Differential Equations, Prentice-Hall, Englewood Cliffs, NJ, 1992.

[15] G. Strang, Linear Algebra and its Application, 2nd ed. Academic Press, New York, 1980.

[16] User's Manual, Math Library version 1.0, IMSL Library edition 10.0, IMSL, Houston, TX, 1987.